# Plate models in elasticity obtained by simultaneous homogenization and dimensional reduction

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# Outline

- Plate models in elasticity
- Homogenization, two scale convergence
- Homogenization and dimensional reduction
- Conclusions

# Plate models in elasticity

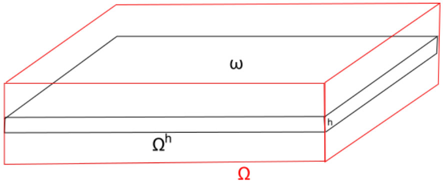
Minimization functional of $3D$ elasticity

$$\int_\Omega W(\nabla \boldsymbol{u})dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx,$$

- $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ deformation
- $\boldsymbol{f} : \Omega \to \mathbb{R}^3$, external volume dead loads
- $W : \mathbb{R}^{3\times 3} \to [0, \infty]$, stored energy function with the properties (important for higher order models)
    1. class $C^2$ in a neighborhood of $\mathrm{SO}(3)$;
    2. $W$ is frame-indifferent, i.e., $W(\mathbf{F}) = W(\mathbf{RF})$ for every $\mathbf{F} \in \mathbb{R}^{3\times 3}$ and $\mathbf{R} \in \mathrm{SO}(3) \Leftrightarrow W(\mathbf{F}) = W(\sqrt{\mathbf{F}^T\mathbf{F}})$.
    3. $W(\mathbf{F}) \geq C_W \,\mathrm{dist}^2(\mathbf{F}, \mathrm{SO}(3))$, for some $C_W > 0$ and all $\mathbf{F} \in \mathbb{R}^{3\times 3}$, $W(\mathbf{F}) = 0$ iff $\mathbf{F} \in \mathrm{SO}(3)$.

# Plate models in elasticity

- $$\int_{\Omega^h} W(\nabla \boldsymbol{u}^h)dx - \int_{\Omega^h} \boldsymbol{f}^h \cdot \boldsymbol{u}^h dx,$$
  $\Omega^h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, $\omega \subset \mathbb{R}^2$ Lipschitz domain.




What happens as $h \to 0$?
Rescaling: $P^h : \Omega \to \Omega^h$, $P^h(x_1, x_2, x_3) = (x_1, x_2, hx_3)$,
$\Omega = \omega \times [-\frac{1}{2}, \frac{1}{2}]$. Minimization functional:

$$\int_{\Omega} W(\nabla_h \boldsymbol{u}^h)dx - \int_{\Omega} \boldsymbol{f}^h \cdot \boldsymbol{u}^h dx,$$

$\nabla_h = \nabla_{\boldsymbol{e}_1, \boldsymbol{e}_2} + \frac{1}{h}\nabla_{\boldsymbol{e}_3}$-physical gradient translated on the canonical domain.

## Plate models in elasticity-Γ-convergence

### Definition (De Giorgi)

*$(X, d)$ metric space. $f^n : X \to \overline{\mathbb{R}}$ Γ-converges to $f : X \to \overline{\mathbb{R}}$*

1. *(*$\liminf$ *inequality) for every sequence $(x^n)_n$ which converges to $x$ we have*

$$f(x) \leq \liminf_n f^n(x^n);$$

2. *(*$\limsup$ *inequality) there exist $(x^n)_n$ (recovery sequence) which converges to $x$ such that*

$$f(x) = \lim_n f^n(x^n).$$

- Γ-limit + precompactness of minimizing sequence $\Longrightarrow$ convergence of global minimizers to minimizer of the limit;
- the non-uniqueness or non-existence of the minimizers does not bother us.
- the limit functional is lower semicontinuous.

## Plate models in elasticity-hierarchy of models

Depending on parameter $\alpha \geq 0$ and boundary conditions (i.e. appropriate space) we want to find (without the term of forces)

$$\frac{1}{h^\alpha}\int_\Omega W(\nabla_h \boldsymbol{u})dx \xrightarrow{\Gamma} ?$$

Goal: To have (and justify) simpler models which will take into the account the thickness.

- $\alpha = 0$ membrane model Le Dret, Raoult (1995).
- $\alpha > 0$ the key fact is the frame-indifference of the stored energy function.

  $\alpha = 2$ (2002) the limit deformations are exact isometries, defined on $\omega$, i.e. $\nabla \boldsymbol{u} \in \mathrm{SO}(3)$ a.e. in $\omega$. The key mathematical tool is the geometric rigidity theorem proved by Friesecke, James, Müller.

# Plate models in elasticity-membrane model

Assumption: free boundary condition, $W : \mathbb{R}^{3\times3} \to \mathbb{R}$, $p$ growth and coercivity assumption

$$\exists C_1, C_2 > 0, C_3 \in \mathbb{R} \text{ s.t. } \forall \mathbf{F} \in \mathbb{R}^{3\times3}$$

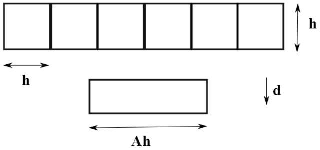
$$C_1|\mathbf{F}|^p - C_3 \leq W(\mathbf{F}) \leq C_2|\mathbf{F}|^p + C_3.$$

- $\alpha = 0$, membrane case.

  compactness result $\nabla_h \boldsymbol{u}^h$ bdd. in $L^p \implies \boldsymbol{u}$ does not depend on $x_3$.

  Compacness result: limit deformations $\boldsymbol{u}(\hat{x})$. Relaxation (first attempt):

$$\boldsymbol{u}^h(\hat{x}, x_3) = \boldsymbol{u}(\hat{x}) + hx_3\boldsymbol{d}(\hat{x}).$$

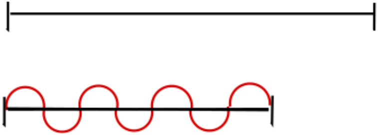


# Plate models in elasticity-membrane model

- i) Relax by $\boldsymbol{d}$. $W_0(\mathbf{G}) = \min_{\boldsymbol{d}\in\mathbb{R}^3} W(\mathbf{G}, \boldsymbol{d})$ (not correct).
  ii) Relax by possible fine oscillations $\Longrightarrow$ quasiconvex envelope

$$QW_0(\mathbf{G}) = \inf_{\psi\in W_0^{1,\infty}(\omega;\mathbb{R}^3)} \int_{[0,1]^2} W_0(\mathbf{G} + \nabla\psi(y))dy.$$

The correct energy functional is $\int_\omega QW_0(\nabla \boldsymbol{u}(\hat{x}))d\hat{x}$.



iii) objectivity $\Longrightarrow$ the energy depends on $(\nabla \boldsymbol{u})^T \nabla \boldsymbol{u}$.

# Plate models in elasticity-bending model

- $\alpha = 2$. Take $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) \in W^{1,2}(\omega; \mathrm{SO}(3))$ such that for $\boldsymbol{u} \in W^{2,2}(\omega; \mathbb{R}^3)$, $\nabla_{\hat{x}} \boldsymbol{u} = (\mathbf{R}_1 | \mathbf{R}_2)$. Define

$$\boldsymbol{u}^h(\hat{x}, x_3) = \boldsymbol{u}(\hat{x}) + h x_3 \mathbf{R}_3(\hat{x}) + O(h^2).$$

$$\nabla_h \boldsymbol{u}^h = \mathbf{R} + O(h),\ (\nabla_h \boldsymbol{u})^T \nabla_h \boldsymbol{u} = \mathbf{I} + O(h),$$

$$W(\nabla_h \boldsymbol{u}^h) = W\Big(\sqrt{(\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h}\Big) = W\big(\mathbf{I} + O(h)\big) = O(h^2).$$

compactness result by TGR (minimizers compact in strong topology).

Theorem (on geometric rigidity, Friesecke, James, Müller)

*Let $U \subset \mathbb{R}^m$ be a bounded Lipschitz domain, $m \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $\boldsymbol{v} \in W^{1,2}(U;\mathbb{R}^m)$ there is associated rotation $\mathbf{R} \in \mathrm{SO}(m)$ such that*

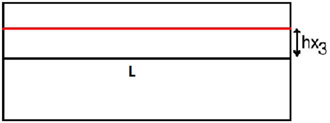
$$\|\nabla \boldsymbol{v} - \mathbf{R}\|_{L^2(U)} \leq C(U)\|\operatorname{dist}(\nabla \boldsymbol{v}, \mathrm{SO}(m))\|_{L^2(U)}.$$

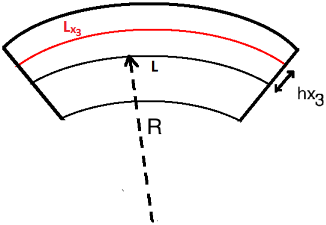
*The constant $C(U)$ can be controlled for the class of Billipschitz domains whose Billipschitz constants we can control.*

# Plate models in elasticity-bending model

- Strain: $(\mathbf{R}^T\partial_1\mathbf{R}_3, \mathbf{R}^T\partial_2\mathbf{R}_3)$ -curvature

$$L_{x_3} = (1 + \frac{hx_3}{R})L$$

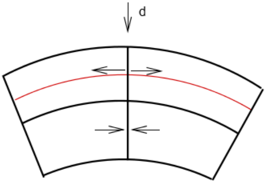


# Plate models in elasticity-bending model

- Relaxation by higher order term

$$\boldsymbol{u}^h(\hat{x}, x_3) = \boldsymbol{u}(\hat{x}) + hx_3\mathbf{R}_3(\hat{x}) + \frac{1}{2}h^2x_3^2\boldsymbol{d}(\hat{x}).$$




Energy density : $Q_2(\mathbf{R}^T\partial_1\mathbf{R}, \mathbf{R}^T\partial_2\mathbf{R})$.

$$Q(\mathbf{G}) = D^2W(\mathbf{I})(\mathbf{G}, \mathbf{G}),\ Q_2(\mathbf{G}) = \min_{\boldsymbol{a}\in\mathbb{R}^3} Q(\mathbf{G}+\boldsymbol{a}\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{a}).$$

Thin bodies (in bending) are (only?) geometrically nonlinear and more universal.

# Plate models in elasticity-von Kármán model

- The "typical deformation" of order $\alpha = 4$ looks like

$$\boldsymbol{u}^h_{\mathrm{vK}} = \begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 \boldsymbol{z} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix} + O(h^2),$$

$\hat{x} = (x_1, x_2)$, $\boldsymbol{z} : \omega \to \mathbb{R}^2$, $v : \omega \to \mathbb{R}$.

$$\begin{aligned} \nabla_h \boldsymbol{u}^h &= \mathbf{I} + h \left( \begin{array}{c|c} 0 & -(\nabla' v)^T \\ \hline \nabla' v & 0 \end{array} \right) + O(h^2) \\ (\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h &= \mathbf{I} + O(h^2). \end{aligned}$$

Taylor expansion of $\tilde{W}$ about $\mathbf{I}$ $\Longrightarrow$

$$W(\nabla_h \boldsymbol{u}^h) = W\left(\sqrt{(\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h}\right) = W\left(\mathbf{I} + O(h^2)\right) = O(h^4).$$

# Plate models in elasticity-hierarchy of models

- The property of objectivity of energy density function influences the process of dimensional reduction causing the hierarchy of models
- For $\alpha \geq 2$ the models are geometrically nonlinear i.e. quadratic in strain which can be nonlinear.
- the only parameter in the limit procedure we change is the order of energy with respect to the thickness.

## Plate models in elasticity-hierarchy of models

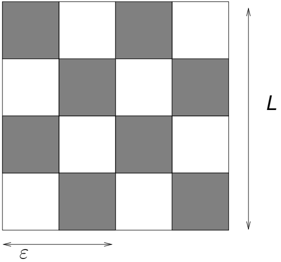
Assumption: free boundary condition

$$\frac{1}{h^\alpha}\int_\Omega W(\nabla_h \boldsymbol{u})dx \xrightarrow{\Gamma} ?$$

- $\alpha = 0$ membrane model;
- $0 < \alpha < 5/3$ trivial limit; zero for short map ($(\nabla \boldsymbol{u})^T \nabla \boldsymbol{u} \leq \mathbf{I}$); infinity for the others;
- $5/3 \leq \alpha < 2$ open;
- $\alpha = 2$ bending model;
- $2 < \alpha < 4$ constrained models;
- $\alpha = 4$ von Kármán model;
- $\alpha > 4$ linear von Kármán theory

Clamped plate: $0 < \alpha < 4$ Föppl displacement theory.
Rod: due to simpler geometry in regime $0 < \alpha < 2$ trivial.

# Homogenization-elliptic equation




$$\Omega = [0, L]^n; u^\varepsilon \in H_0^1(\Omega)$$

$$-\operatorname{div}(\mathbf{A}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_i}) = f$$

$$\mathbf{A}^\varepsilon(x) = \mathbf{A}_{ij}(\frac{x}{\varepsilon}), \quad \mathbf{A}_{ij} \text{ periodic on } Y := [0,1]^n, \text{ standard conditions}$$

Goal $u^\varepsilon \overset{?}{\to} u_0$; the equation for $u_0$? $\mathbf{A}^\varepsilon \overset{*}{\rightharpoonup} \int_Y \mathbf{A}(y)dy$. Are the coeffficients of the limit equation weak limits of the coefficients?

# Homogenization-elliptic equation

1. formal asymptotics

$$u^\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots ;$$

$$u_0, u_1 \ldots \text{ periodic in } y$$

2. conclude $u_0$ is dependent only on "slow variable" $x$. Identify $u_0$ and $u_1$.
3. Prove the convergence result (e.g. Tartar method of oscillating functions).

## Homogenization-two scale convergence

$\Omega \subset \mathbb{R}^n$; $p \in \langle 1, +\infty \rangle$.

### Definition (Nguetseng, Allaire)

A bounded sequence in $L^p(\Omega) \ni u^\varepsilon \overset{2}{\rightharpoonup} u_0 \in L^p(\Omega \times Y)$ if for every smooth $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which is $Y$ periodic in the second variable we have

$$\int_\omega u^\varepsilon(x)\psi(x, \frac{x}{\varepsilon}) \to \int_\omega \int_Y u_0(x, y)\psi(x, y) \quad \text{as } \varepsilon \to 0.$$

Remark: Unlike in Young measures, we are interested in the "speed of oscillations". It gives us additional information about weak convergent sequences:

$$u^\varepsilon \overset{2}{\rightharpoonup} u_0 \implies u^\varepsilon \rightharpoonup \int_Y u_0(x, y)dy.$$

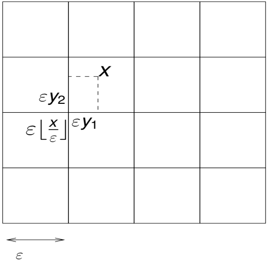
# Homogenization-two scale convergence

$\mathcal{Y} = [0, 1\rangle^n$ with the topology of torrus

- Define $S^\varepsilon : \mathbb{R}^n \times \mathcal{Y} \to \mathbb{R}^n$

$$S^\varepsilon(x, y) := \varepsilon\lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y$$

$$u^\varepsilon \overset{2}{\rightharpoonup} u_0 \Leftrightarrow u^\varepsilon \circ S^\varepsilon \rightharpoonup u_0 \text{ in } L^p(\mathbb{R}^n \times \mathcal{Y}).$$



# Homogenization-two scale convergence

- compactness result: $p > 1$,

$$u^\varepsilon \text{ bounded in } L^p \implies \exists \text{ subsequence and } u_0 \in L^p(\Omega \times \mathcal{Y}), u^\varepsilon \xrightarrow{2} u_0$$

- Example: $f \in L^p(\mathbb{R}^n; C(\mathcal{Y}))$ ; $u^\varepsilon = f(x, \frac{x}{\delta(\varepsilon)})$

$$u^\varepsilon \xrightarrow{2} \begin{cases} \int_Y f(x,y)dy, & \text{if } \lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\varepsilon} = 0 \text{ or } \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0 \\ f(x,y), & \text{if } \delta(\varepsilon) = \varepsilon \end{cases}$$

-

$$u^\varepsilon \text{ bounded in } W^{1,p}(\Omega) \Rightarrow \exists u_0 \in W^{1,p}(\Omega), u_1 \in L^p(\Omega; W^{1,p}(\mathcal{Y}))$$

$$\text{s.t. } \nabla u^\varepsilon \xrightarrow{2} \nabla u_0 + \nabla_y u_1(x,y)$$

-

$$u^\varepsilon \xrightarrow{2} u \Rightarrow \liminf_{\varepsilon\to 0} \|u^\varepsilon\|_{L^p} \geq \|u\|_{L^p(\mathbb{R}^n \times \mathcal{Y})} \geq \left\| \int_Y u(\cdot, y)dy \right\|_{L^p(\mathbb{R}^n)}$$

# Homogenization-two scale convergence

- $p \in \langle 1, +\infty \rangle$; $\Omega \subset \mathbb{R}^n$; $W : \Omega \times \mathcal{Y} \times \mathbb{R}^N \to \mathbb{R}$ Charathéodory function, bdd from below; $W(x, y, \cdot)$ convex and lsc

$$I^\varepsilon = \int_\Omega W(x, \tfrac{x}{\varepsilon}, \boldsymbol{u}(x))dx$$

is lower semicontinuos wrt two scale convergence i.e.

$$\boldsymbol{u}^\varepsilon \overset{2}{\rightharpoonup} \boldsymbol{u}_0 \text{ in } L^p(\Omega \times \mathcal{Y})^N$$

$$\Rightarrow \liminf I^\varepsilon(\boldsymbol{u}^\varepsilon) \geq I(\mathbf{u}) := \iint_{\Omega \times \mathcal{Y}} W(x, y, \boldsymbol{u}(x, y))dydx$$

# Homogenization-convex integrands

## Theorem (Marcellini,1978)

*$\Omega \subset \mathbb{R}^n$, domain, $W : \mathcal{Y} \times \mathbb{R}^{n\times m} \to \mathbb{R}$ convex with p-growth and coercivity assumption*

$$\exists C_1, C_2 > 0, C_3 \in \mathbb{R} \; s.t. \; \forall y \in \mathcal{Y}, \forall \mathbf{F} \in \mathbb{R}^{n\times m}$$

$$C_1|\mathbf{F}|^p - C_3 \leq W(y, \mathbf{F}) \leq C_2|\mathbf{F}|^p + C_3.$$

*Define:*

$$I^\varepsilon(\boldsymbol{u}) = \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla \boldsymbol{u}(x))dx, \quad I(\boldsymbol{u}) = \int_\Omega W_{hom}(\nabla \boldsymbol{u}(x))dx$$

$$W_{hom}(\mathbf{F}) = \inf_{\psi \in W^{1,p}(\mathcal{Y};\mathbb{R}^m)} \int_{\mathcal{Y}} W(y, \mathbf{F} + \nabla\psi(y))dy$$

*Then*

$$I^\varepsilon \xrightarrow{\Gamma} I \; in \; W^{1,p}(\Omega; \mathbb{R}^m)$$

# Homogenization-convex integrands

## Proof.

1. liminf-inequality: take minimizing sequence $\boldsymbol{u}^\varepsilon$. Then for a subsequence it is valid

$$\exists \boldsymbol{u}_0 \in W^{1,p}(\Omega;\mathbb{R}^m),\ \boldsymbol{u}_1 \in L^p(\Omega; W^{1,p}(\Omega;\mathbb{R}^m))$$

$$\text{s.t. } \nabla \boldsymbol{u}^\varepsilon \overset{2}{\rightarrow} \nabla \boldsymbol{u}_0 + \nabla_y \boldsymbol{u}_1(x,y)$$

Due to lower semicontinuity we can conclude:

$$\int_\Omega W(\tfrac{x}{\varepsilon}, \nabla \boldsymbol{u}^\varepsilon)dx \geq \int_\Omega \int_Y W\big(y, \nabla \boldsymbol{u}_0(x) + \nabla \boldsymbol{u}_1(x,y)\big)dydx$$

$$\geq \int_\Omega W_{\mathrm{hom}}(\nabla \boldsymbol{u}_0(x))dx.$$

2. recovery sequence: direct; of the type $\boldsymbol{u}_0(x) + \varepsilon \boldsymbol{u}_1(x, \frac{x}{\varepsilon})$.

□

# Homogenization-nonconvex integrands

Remark: In the convex case differential equations for the corrector are available. The profile of the oscillations can be obtained by single cell formula.

Theorem (Braides,Müller)

*$W$ not necesseraly convex; satisfies $p$ growth and coercivity condition. The multiple cell formula is given by*

$$W_{hom}(\mathbf{F}) = \inf_{k\in\mathbb{N}} \inf_{\psi\in W_0^{1,p}(kY;\mathbb{R}^m)} \frac{1}{k^n}\int_{kY} W\big(y, \mathbf{F} + \nabla\psi(y)\big)\, dy.$$

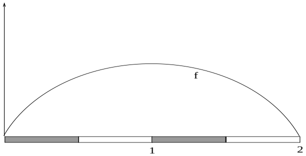
Remark: No equations for the corrector are available.

## Homogenization- comparison between convex and nonconvex integrands

- unimportance of the "boundary conditions" (i.e. we can put periodic or zero boundary conditions)
- for convex functions

$$\inf_{k\in\mathbb{N}} \inf_{\psi\in W^{1,p}_{\text{per}}(kY;\mathbb{R}^m)} \frac{1}{k^n}\int_{kY} W\big(y,\mathbf{F}+\nabla\psi(y)\big)\,dy =$$

$$\inf_{\psi\in W^{1,p}(\mathcal{Y};\mathbb{R}^m)} \int_{\mathcal{Y}} W\big(y,\mathbf{F}+\nabla\psi(y)\big)\,dy$$




$\boldsymbol{f} : [0,2] \to \mathbb{R}^m$ periodic. Define $\boldsymbol{g} : [0,1] \to \mathbb{R}^m$ periodic

$$\boldsymbol{g}(y) = \frac{\boldsymbol{f}(y)+\boldsymbol{f}(y+1)}{2}.$$

# Homogenization- comparison between convex and nonconvex integrands

- $\mathbf{F} \in \mathbb{R}^m$ and $W$ convex one periodic:

$$\int_Y W(y, \mathbf{F} + \nabla \boldsymbol{g}(y))dy \leq$$

$$\frac{1}{2}\int_Y W(y, \mathbf{F} + \nabla \boldsymbol{f}(y))dy + \frac{1}{2}\int_Y W(y, \mathbf{F} + \nabla \boldsymbol{f}(y+1))dy$$

$$= \frac{1}{2}\int_{2Y} W(y, \mathbf{F} + \nabla \boldsymbol{f}(y))dy.$$

## Homogenization and dimensional reduction-membrane case of plate


Assumption: $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma \in \langle 0, \infty\rangle$.

$\omega \subset \mathbb{R}^2$ domain; $\Omega = \omega \times [-\frac{1}{2}, \frac{1}{2}]$; $\Omega^h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$

### Theorem (Braides, Fonseca, Francfort)

*$\Omega$ domain, $Y = [0,1]^2$, $\mathcal{Y} = [0,1\rangle^2$-torrus, $W : \mathcal{Y} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ continuous with p growth and coercivity assumption*

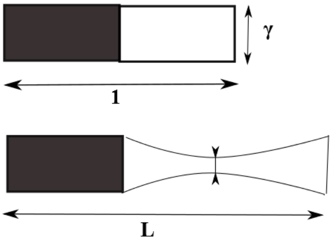
$$I^h(\boldsymbol{u}) = \int_\Omega W\left(\frac{x_\alpha}{\varepsilon(h)}, \nabla_h \boldsymbol{u}\right)dx \xrightarrow[W^{1,p}]{\Gamma} I(\boldsymbol{u}) = \int_\omega W_{hom}(\nabla \boldsymbol{u})d\hat{x}$$

*Define $\mathcal{S}_k = \{\psi \in W^{1,p}(kY \times I; \mathbb{R}^3) : \psi = 0 \text{ if } x_\alpha = 0,\ \alpha = 1,2\}$. The unique homogenization-dimensional reduction multiple cell formula is given by:*

$$W_{hom}(\mathbf{G}) = \inf_{k\in\mathbb{N}} \inf_{\psi\in\mathcal{S}_k} \left\{ \frac{1}{k^2} \iint_{kY\times I} W\left(y, \mathbf{G} + \left(\nabla_y \psi | \tfrac{1}{\gamma}\partial_{x_3}\psi\right)\right) dydx_3 \right\}$$

## Homogenization and dimensional reduction-membrane case of plate

Relaxation: $\mathbf{G}\hat{x} + k\varepsilon(h)\psi(\hat{x}, \frac{\hat{x}}{k\varepsilon(h)}, \frac{x_3}{\gamma})$, e.g. $k = 1$



# Homogenization and dimensional reduction-bending case of plate

Non-oscillating material:

$$\nabla_h \boldsymbol{u}^h \approx \mathbf{R}^h + h\,\mathrm{rest} \implies (\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h \approx \mathbf{I} + h\,\mathrm{rest},$$

$$\frac{1}{h^2} W(\nabla_h \boldsymbol{u}^h) = \frac{1}{h^2} W\Big(\sqrt{(\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h}\Big) \approx Q(\mathrm{rest}),$$

where $Q(\mathbf{G}) = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\mathbf{I})[\mathbf{G}, \mathbf{G}]$. $Q$ is positive semidefinite quadratic form, strictly convex on symmetrix matrices.
Is two scale analysis (on the level of the material) enough to capture the homogenization effects in bending regime?
In the case of rods the result is given by Neukamm (ARMA, 2012) where he uses the simple geometry of rod. Only two scale analysis is enough...

# Homogenization and dimensional reduction-bending case of plate


Assumption: $\lim_{h \to 0} \frac{h}{\varepsilon(h)} = \gamma \in \langle 0, \infty \rangle$.

Procedure for Γ-limit

- compactness result (what can we say about the deformations whose energy is of order $h^2$?)
- liminf inequality
- recovery sequence

## Homogenization and dimensional reduction-bending case of plate

What can we extract from compactness result?

$$e^h(\boldsymbol{u}^h) = \frac{1}{h^2}\int_\Omega \mathrm{dist}^2(\nabla_h \boldsymbol{u}^h, \mathrm{SO}\,3)dx.$$

Divide $\Omega$ in cubes $\varepsilon(h) \times \varepsilon(h) \times [-\frac{1}{2}, \frac{1}{2}]$ which follow the structure of the material.
(TGR) $\implies$ there exist $\mathbf{R}^h : \omega \to \mathrm{SO}\,3$, piecewise constant on cubes $\varepsilon(h) \times \varepsilon(h) \times [-\frac{1}{2}, \frac{1}{2}]$, and $\mathbf{P}^h \in W^{1,2}(\omega; \mathbf{M}^3)$ such that:

$$\begin{aligned}
\|\nabla_h \boldsymbol{u}^h - \mathbf{R}^h\|^2_{L^2(\Omega)} &\lesssim h^2 e_h(\boldsymbol{u}^h), \\
\|\nabla_h \boldsymbol{u}^h - \mathbf{P}^h\|^2_{L^2(\Omega)} &\lesssim h^2 e_h(\boldsymbol{u}^h), \\
\|\mathbf{P}^h - \mathbf{R}^h\|^2_{L^2(\omega)} &\lesssim h^2 e_h(\boldsymbol{u}^h), \\
\|\hat{\nabla}\mathbf{P}^h\|^2_{L^2(\omega)} &\lesssim e_h(\boldsymbol{u}^h), \\
\|\mathbf{P}^h\|^2_{L^\infty(\omega)} &\lesssim 1,
\end{aligned}$$

# Homogenization and dimensional reduction-bending case of plate

## Theorem (Hornung, Neukamm, V.)

*$\omega$-convex*, $\Omega = \omega \times [-\frac{1}{2}, \frac{1}{2}]$. *Under the standard assumptions on $W$ ($W : \mathcal{Y} \times \mathbb{R}^{3\times 3} \to \mathbb{R}^+$ periodic in the first variable, objective and uniform quadratic lower bound away from $\mathrm{SO}(3)$ ) we have*

$$I^h(\boldsymbol{u}) = \frac{1}{h^2} \int_\Omega W(\tfrac{x_\alpha}{\varepsilon(h)}, \nabla_h \boldsymbol{u}) dx \xrightarrow[H^1]{\Gamma} I(\boldsymbol{u}),$$

*where*

$$I(\boldsymbol{u}) = \begin{cases} \int_\omega Q^\gamma_{hom}(\, II_{(\boldsymbol{u})}) dx, & \textit{if } \boldsymbol{u} \in W^{2,2}(\bar{\omega}; \mathbb{R}^3) \textit{ isometry} \\ +\infty, & \textit{otherwise} \end{cases}$$

*$Q^\gamma_{hom}$ preserves the properties of $Q$ i.e. it is quadratic and positive definite on symmetric matrices.*

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Here $Q^{\gamma}_{\mathrm{hom}} : \mathbf{M}^2 \to \mathbb{R}^+$ is defined by

$$Q^{\gamma}_{\mathrm{hom}}(\boldsymbol{A}) := \inf_{\substack{\phi \in H^1(I \times \mathcal{Y}; \mathbb{R}^3) \\ \mathbf{B} \in \mathbf{M}^2_{\mathrm{sym}}}} \iint_{I \times Y} Q\left(y,\ \Lambda(\boldsymbol{A}, \boldsymbol{B}) + (\hat{\nabla}_y \phi \mid \tfrac{1}{\gamma}\partial_3 \phi)\right) dy dx_3$$

# Plate models in elasticity-von Kármán model

- The "typical deformation" of order $\alpha = 4$ looks like

$$\boldsymbol{u}^h_{\mathrm{vK}} = \begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2\boldsymbol{z} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix} + O(h^2),$$

$\hat{x} = (x_1, x_2)$, $\boldsymbol{z} : \omega \to \mathbb{R}^2$, $v : \omega \to \mathbb{R}$.

$$\begin{aligned} \nabla_h \boldsymbol{u}^h &= \mathbf{I} + h \left( \begin{array}{c|c} 0 & -(\nabla' v)^T \\ \hline \nabla' v & 0 \end{array} \right) + O(h^2) \\ (\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h &= \mathbf{I} + O(h^2). \end{aligned}$$

Taylor expansion of $\tilde{W}$ about $\mathbf{I}$ $\Longrightarrow$

$$W(\nabla_h \boldsymbol{u}^h) = W\left(\sqrt{(\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h}\right) = \tilde{W}\left(\mathbf{I} + O(h^2)\right) = O(h^4).$$

# Plate models in elasticity-von Kármán model

- From TGR we conclude that the limit deformation is rigid deformation and we are in the displacement theory (after changing the coordinates).

$$\int_\Omega \mathrm{dist}^2(\nabla_h \boldsymbol{u}^h, \mathrm{SO}(3))dx \leq Ch^4 \overset{\mathrm{TGR}}{\Longrightarrow} \text{ on a subsequence}$$

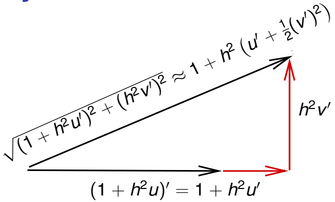
$$\boldsymbol{z}^h := \frac{1}{h^2}\left( \int_I \begin{pmatrix} \boldsymbol{u}_1^h \\ \boldsymbol{u}_2^h \end{pmatrix} dx_3 - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \overset{H^1}{\rightharpoonup} \boldsymbol{z}$$

$$v^h := \frac{1}{h}\int_I \boldsymbol{u}_3^h dx_3 \overset{H^1}{\to} v, \quad v \in H^2$$

$$\text{(strain) } \mathbf{G}^h = \frac{\sqrt{(\nabla_h \boldsymbol{u}^h)^T \nabla_h \boldsymbol{u}^h} - \mathbf{I}}{h^2} \overset{L^2}{\rightharpoonup}$$

$$\mathrm{sym}(\nabla \boldsymbol{z}) + \frac{1}{2}\mathrm{sym}(\nabla v \otimes \nabla v) - x_3 \nabla^2 v$$

# Plate models in elasticity-von Kármán model




$$\mathbf{G} = \mathbf{G}_1 - x_3\mathbf{G}_2$$

$$\mathbf{G}_1 = \mathrm{sym}(\nabla \boldsymbol{z}) + \frac{1}{2}\,\mathrm{sym}(\nabla v \otimes \nabla v),\ \mathbf{G}_2 = \nabla^2 v$$

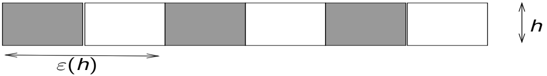
$\mathbf{G}_1$- stretching of the central plane;
$x_3\mathbf{G}_2$- corrector for the stretching of the plane at level $hx_3$.

Energy density: $Q_2(\mathbf{G}_1) + \frac{1}{12}Q_2(\mathbf{G}_2)$

Relaxation by higher order terms:

$$\boldsymbol{u}^h = \boldsymbol{u}_{\mathrm{vK}} + h^3x_3\boldsymbol{d}_1(\hat{x}) + h^3x_3^2\boldsymbol{d}_2(\hat{x}).$$

# Homogenization and dimensional reduction-von Kármán plate




Assumption: $\lim_{h\to 0} \frac{h}{\varepsilon(h)} = \gamma \in \langle 0, \infty \rangle$.

$$\boldsymbol{u}^h_{\mathrm{vK}} = \underbrace{\begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 \boldsymbol{z} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix}}_{\text{compactness}}$$

$$+ \underbrace{h^3 x_3 \boldsymbol{d}_1 + h^3 x_3^2 \boldsymbol{d}_2}_{\text{relaxation}}$$

$$\boldsymbol{u}^h_{\mathrm{HvK}} = \begin{pmatrix} \hat{x} \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 \boldsymbol{z} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix}$$

$$+ h^2 \varepsilon(h) \psi(x_1, x_2, \tfrac{x_1}{\varepsilon(h)}, \tfrac{x_2}{\varepsilon(h)}, x_3)$$

## Homogenization and dimensional reduction-von Kármán plate

### Theorem (Neukamm,V.)

*Under the standard assumptions on W*

$$I^h(\boldsymbol{u}) = \frac{1}{h^4}\int_\Omega W\left(\tfrac{x_\alpha}{\varepsilon(h)}, \nabla_h \boldsymbol{u}\right)dx \xrightarrow[H^1]{\Gamma} I(\boldsymbol{z}, v) = \int_\omega Q^\gamma_{hom}(\mathbf{G}_1, \mathbf{G}_2)dx,$$

*where*

$$Q^\gamma_{hom}(\boldsymbol{A}, \boldsymbol{B}) :=$$

$$\inf_{\psi \in H^1(I\times\mathcal{Y},\mathbb{R}^3)} \iint_{I\times\mathcal{Y}} Q\left(y,\ \boldsymbol{A} - x_3\boldsymbol{B} + (\hat{\nabla}_y\psi, \tfrac{1}{\gamma}\partial_{x_3}\psi)\right) dydx_3,$$

$$I = [-\tfrac{1}{2}, \tfrac{1}{2}], \quad Q(y, \mathbf{G}) = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\mathbf{I})[\mathbf{G}, \mathbf{G}], \mathbf{G} \in \mathbb{R}^{3\times 3}$$

$Q^\gamma_{hom}$ *quadratic, positive definite on symmetric matrices*

Remark:

The case $\gamma = 0$ corresponds to the situation first homogenize then do the dimensional reduction; the case $\gamma = \infty$ corresponds to the situation first do the dimensional reduction then homogenize. Both of these cases can be obtained by taking the limit $\gamma \to 0$ i.e. $\gamma \to \infty$. The quadratic forms $Q_\gamma$ are continuous in $\gamma$ for $\gamma \in [0, \infty]$.

## Homogenization and dimensional reduction-bending and von Kármán plate

- for Gamma limit it is necessary to prove liminf inequality and to find recovery sequence. In general, both can be demanding tasks. The compactness result tells us how the limit deformations look like ("physics" of the problem) and then we do relaxation. Compactness result can be nontrivial and sometimes requires new meningfull functional spaces to be defined (von Kármán shell).
- In the bending case it was nontrivial to find recovery sequence and to prove liminf inequality. Recovery sequence imposed unnatural condition on $\omega$ i.e. convexity. We hope to overcome this condition in the future.
- in the von Kármán case it was not difficult to guess the recovery sequence, but liminf inequality requires some modifications of the "standard techniques".

## Conclusions

- We have given an overview for hierarchy of plate models obtained by simultaneous homogenization and dimensional reduction from $3D$ elasticity. The assumption was that the oscillations are on the same scale as the thickness of the body. The obtained models depend on parameter $\gamma = \lim_{h \to 0} \frac{h}{\varepsilon(h)}$ and can not be obtained from existing lower dimensional models.
- For regimes of order $\alpha \geq 2$, due to essential convexity of energy (in strain), to relax the energy we used the oscillations of the same type as the oscillations of material. The obtained model is significally simpler than the full $3D$ model.
- For bending regime the problems occur due to geometrical diffuculties (working with the set of isometries). These difficulties do not occur in von Kármán case due to the fact that the typical deformations are infinitesimal isometries, which are the linearizations of exact isometries.